Involutive orbits of non-Noether symmetry groups

George Chavchanidze

Department of Theoretical Physics, A. Razmadze Institute of Mathematics, 1 Aleksidze Street, Tbilisi 0193, Georgia

Abstract

Abstract. We consider set of functions on Poisson manifold related by continues one-parameter group of transformations. Class of vector fields that produce involutive families of functions is investigated and relationship between these vector fields and non-Noether symmetries of Hamiltonian dynamical systems is outlined. Theory is illustrated with sample models: modified Boussinesq system and Broer-Kaup system.

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In Hamiltonian integrable models, conservation laws often form involutive orbit of one-parameter symmetry group. Such a symmetry carries important information about integrable model and its bi-Hamiltonian structure. The present paper is an attempt to describe class of one-parameter group of transformations of Poisson manifold that possess involutive orbits and may be related to Hamiltonian integrable systems.
Let $C^\infty(M)$ be algebra of smooth functions on manifold M equipped with Poisson bracket

$$\{f, g\} = W(df \wedge dg) \tag{1}$$

where W is Poisson bivector satisfying property $[W, W] = 0$. Each vector field E on manifold M gives rise to one-parameter group of transformations of $C^\infty(M)$ algebra

$$g_z = e^{zL_E} \tag{2}$$

where L_E denotes Lie derivative along the vector field E. To any smooth function $J \in C^\infty(M)$ this group assigns orbit that goes through J

$$J(z) = g_z(J) = e^{zL_E}(J) = J + zL_E J + \frac{1}{2}z^2(L_E)^2 J + \cdots \tag{3}$$

the orbit $J(z)$ is called involutive if

$$\{J(x), J(y)\} = 0 \qquad \forall x, y \in R \tag{4}$$

Involutive orbits are often related to integrable models where $J(z)$ plays the role of involutive family of conservation laws.
Involutivity of orbit $J(z)$ depends on nature of vector field E and function $J = J(0)$ and in general it is hard to describe all pairs (E, J) that produce involutive orbits however one interesting class of involutive orbits can be outlined by the following theorem:
Theorem 1. For any non-Poisson $[E, W] \neq 0$ vector field E satisfying property

$$[E, [E, W]] = 0 \tag{5}$$

and any function J such that

$$W(dL_E J) = c[E, W](dJ) \qquad c \in R \setminus (0 \cup N) \tag{6}$$

one-parameter family of functions $J(z) = e^{zL_E}(J)$ is involutive.
Proof. By taking Lie derivative of property (6) along the vector field E we get

$$[E, W](dL_E J) + W(d(L_E)^2 J) = c[E, [E, W]](dJ) + c[E, W](dL_E J) \tag{7}$$

where c is real constant which is neither zero nor positive integer. Taking into account (5) one can rewrite result as follows

$$W(d(L_E)^2 J) = (c - 1)[E, W](dL_E J) \tag{8}$$

that after m iterations produces

$$W(d(L_E)^{m+1}J) = (c-m)[E,W](d(L_E)^m J) \tag{9}$$

Now using this property let us prove that functions $J^{(m)} = (L_E)^m J$ are in involution. Indeed

$$\{J^{(k)}, J^{(m)}\} = W(dJ^{(k)} \wedge dJ^{(m)}) \tag{10}$$

Suppose that $k > m$ and let us rewrite Poisson bracket as follows

$$\begin{aligned} &W(dJ^{(k)} \wedge dJ^{(m)}) = W(d(L_E)^k J \wedge dJ^{(m)}) = L_{W(d(L_E)^k J)} J^{(m)} \\ &= (c-k+1) L_{[E,W](d(L_E)^{k-1}J)} J^{(m)} = (c-k+1)[E,W](dJ^{(k-1)} \wedge dJ^{(m)}) \\ &= -(c-k+1) L_{[E,W](d(L_E)^m J)} J^{(k-1)} = -\frac{c-k+1}{c-m} L_{W(d(L_E)^{m+1}J)} J^{(k-1)} \\ &= \frac{c-k+1}{c-m} W(dJ^{(k-1)} \wedge dJ^{(m+1)}) \end{aligned} \tag{11}$$

Thus we have

$$(c-m)\{J^{(k)}, J^{(m)}\} = (c-k+1)\{J^{(k-1)}, J^{(m+1)}\} \tag{12}$$

Using this property $2(m-k)$ times produces

$$\{J^{(k)}, J^{(m)}\} = \{J^{(m)}, J^{(k)}\} \tag{13}$$

and since Poisson bracket is skew-symmetric we finally get

$$\{J^{(k)}, J^{(m)}\} = 0 \tag{14}$$

So we showed that functions $J^{(m)} = (L_E)^m J$ are in involution. In the same time orbit $J(z)$ is linear combination of functions $J^{(m)}$ and thus it is involutive as well.

Remark. Property (9) implies that vector field

$$S = (c-m)E + t(c-m+1)W(dJ^{(m+1)}) \tag{15}$$

is non-Noether symmetry [1] of Hamiltonian dynamical system

$$\frac{d}{dt} f = \{J^{(m)}, f\} \tag{16}$$

in other words non-Poisson vector field S commutes with time evolution defined by Hamiltonian vector field

$$X = \frac{\partial}{\partial t} + W(dJ^{(m)}) \tag{17}$$

This fact can be checked directly

$$\begin{aligned}
&[S, X] = (c-m)[E, X] + t(c-m+1)[W(dJ^{(m+1)}), W(dJ^{(m)})] \\
&-(c-m+1)W(dJ^{(m+1)}) = (c-m)[E, W](dJ^{(m)}) + (c-m)W(dL_E J^{(m)}) \\
&+t(c-m+1)W(d\{J^{(m+1)}, J^{(m)}\}) - (c-m+1)W(dJ^{(m+1)}) \\
&= W(dJ^{(m+1)}) + (c-m)W(dJ^{(m+1)}) - (c-m+1)W(dJ^{(m+1)}) = 0 \quad (18)
\end{aligned}$$

In the same time property (9) means that functions $J^{(m)} = (L_E)^m J$ form Lenard scheme with respect to bi-Hamiltonian structure formed by Poisson bivector fields W and $[E, W]$ (see [1],[4]).

In many infinite dimensional integrable Hamiltonian systems Poisson bivector has nontrivial kernel, and set of conservation laws belongs to orbit of non-Noether symmetry group that goes through centre of Poisson algebra. This fact is reflected in the following theorem:

Theorem 2. If non-Poisson vector field E satisfies property

$$[E, [E, W]] = 0 \tag{19}$$

then every orbit derived from centre I of Poisson algebra $C^\infty(M)$ is involutive.

Proof. If function J belongs to centre $J \in I$ of Poisson algebra $C^\infty(M)$ then by definition $W(dJ) = 0$. By taking Lie derivative of this condition along vector field E one gets

$$W(dL_E J) = -[E, W](dJ) \tag{20}$$

that according to Theorem 1 ensures involutivity of $J(z)$ orbit.

Sample. The theorems proved above may have interesting applications in theory of infinite dimensional Hamiltonian models where they provide simple way to construct involutive family of conservation laws. One non-trivial example of such a model is modified Boussinesq system [2],[5],[6] described by the following set of partial differential equations

$$\begin{aligned}
u_t &= cv_{xx} + u_x v + uv_x \\
v_t &= -cu_{xx} + uu_x + 3vv_x
\end{aligned} \tag{21}$$

where $u = u(x,t), v = v(x,t)$ are smooth functions on R^2 subjected to zero boundary conditions $u(\pm\infty, t) = v(\pm\infty, t) = 0$ This system can be rewritten in Hamiltonian form

$$\frac{d}{dt} f = \{h, f\} = W(dh \wedge df) \tag{22}$$

with the following Hamiltonian

$$h = \frac{1}{2}\int_{-\infty}^{+\infty} (u^2 v + v^3 + 2cuv_x)dx \tag{23}$$

and Poisson bracket defined by Poisson bivector field

$$W = \int_{-\infty}^{+\infty} \frac{1}{2}(A \wedge A_x + B \wedge B_x)dx \tag{24}$$

where A, B are vector fields that for every smooth functional $R = R(u, x)$ are defined via variational derivatives $A(R) = \delta R/\delta u$ and $B(R) = \delta R/\delta v$. For Poisson bivector (24) there exist vector field E such that

$$[E, [E, W]] = 0 \tag{25}$$

this vector field has the following form

$$\begin{aligned} E &= \int_{-\infty}^{+\infty} (uvA_x - cvA_{xx} + (uu_x + vv_x)B + (u^2 + 2v^2)B_x + cuB_{xx})xdx \\ &= -\int_{-\infty}^{+\infty} [(uv + 2cv_x + x((uv)_x + cv_{xx}))A \\ &+ (u^2 + 2v^2 - 2cu_x + x(uu_x + 3vv_x - cu_{xx}))B]dx \end{aligned} \tag{26}$$

Applying one-parameter group of transformations generated by this vector field to centre of Poisson algebra which in our case is formed by functional

$$J = \int_{-\infty}^{+\infty} (ku + mv)dx \tag{27}$$

where k, m are arbitrary constants, produces involutive orbit that recovers infinite sequence of conservation laws of modified Boussinesq hierarchy

$$J^{(0)} = \int_{-\infty}^{+\infty} (ku + mv)dx$$

$$J^{(1)} = L_E J^{(0)} = \frac{m}{2}\int_{-\infty}^{+\infty}(u^2+v^2)dx$$
$$J^{(2)} = (L_E)^2 J^{(0)} = m\int_{-\infty}^{+\infty}(u^2 v + v^3 + 2cuv_x)dx$$
$$J^{(3)} = (L_E)^3 J^{(0)} = \frac{3m}{4}\int_{-\infty}^{+\infty}(u^4 + 5v^4 + 6u^2v^2$$
$$-12cv^2u_x + 4c^2u_x^2 + 4c^2v_x^2)dx$$
$$J^{(m)} = (L_E)^m J^{(0)} = L_E J^{(m-1)} \tag{28}$$

Sample. Another interesting model that has infinite sequence of conservation laws lying on single orbit of non-Noether symmetry group is Broer-Kaup system [3],[5],[6], or more precisely special case of Broer-Kaup system formed by the following partial differential equations

$$\begin{aligned} u_t &= cu_{xx} + 2uu_x \\ v_t &= -cv_{xx} + 2uv_x + 2u_x v \end{aligned} \tag{29}$$

where $u = u(x,t), v = v(x,t)$ are again smooth functions on R^2 subjected to zero boundary conditions $u(\pm\infty, t) = v(\pm\infty, t) = 0$ Equations (29) can be rewritten in Hamiltonian form

$$\frac{d}{dt}f = \{h, f\} = W(dh \wedge df) \tag{30}$$

with the Hamiltonian equal to

$$h = \int_{-\infty}^{+\infty}(u^2v + cu_x v)dx \tag{31}$$

and Poisson bracket defined by

$$W = \int_{-\infty}^{+\infty} A \wedge B_x dx \tag{32}$$

One can show that the following vector field E

$$E = \int_{-\infty}^{+\infty}(u^2A_x - cuA_{xx} + (uv)_x B + 3uvB_x + cvB_{xx})xdx$$
$$= -\int_{-\infty}^{+\infty}[(u^2 + 2cu_x + x(2uu_x + cu_{xx}))A$$
$$+(3uv - 2cv_x + x(2(uv)_x - cv_{xx}))B]dx \tag{33}$$

has property

$$[E, [E, W]] = 0 \tag{34}$$

and thus group of transformations generated by this vector field transforms centre of Poisson algebra formed by functional

$$J = \int_{-\infty}^{+\infty} (ku + mv)dx \tag{35}$$

into involutive orbit that reproduces well known infinite set of conservation laws of modified Broer-Kaup hierarchy

$$\begin{aligned}
J^{(0)} &= \int_{-\infty}^{+\infty} (ku + mv)dx \\
J^{(1)} &= L_E J^{(0)} = m \int_{-\infty}^{+\infty} uvdx \\
J^{(2)} &= (L_E)^2 J^{(0)} = 2m \int_{-\infty}^{+\infty} (u^2 v + cu_x v)dx \\
J^{(3)} &= (L_E)^3 J^{(0)} = 3m \int_{-\infty}^{+\infty} (2u^3 v - 3cu^2 v_x - 2c^2 u_x v_x)dx \\
J^{(m)} &= (L_E)^m J^{(0)} = L_E J^{(m-1)}
\end{aligned} \tag{36}$$

Two samples discussed above are representatives of one interesting family of infinite dimensional Hamiltonian systems formed by D partial differential equations of the following type

$$\begin{aligned}
&U_t = -2FGU_{xx} + \langle U, GU_x \rangle C + \langle C, GU_x \rangle U + \langle C, GU \rangle U_x \\
&detG \neq 0, \qquad G^T = G, \qquad F^T = -F \\
&F_{mn}C_k + F_{km}C_n + F_{nk}C_m = 0
\end{aligned} \tag{37}$$

where U is vector with components u_m that are smooth functions on R^2 subjected to zero boundary conditions

$$u_m = u_m(x, t); \qquad u_m(\pm\infty, t) = 0; \qquad m = 1...D \tag{38}$$

G is constant symmetric nondegenerate matrix, F is constant skew-symmetric matrix, C is constants vector that satisfies condition

$$F_{mn}C_k + F_{km}C_n + F_{nk}C_m = 0 \tag{39}$$

and $\langle , \rangle$ denotes scalar product

$$\langle X, Y \rangle = \sum_{m=1}^{D} X_m Y_m. \tag{40}$$

System of equations (37) is Hamiltonian with respect to Poisson bivector equal to

$$W = \int_{-\infty}^{+\infty} \langle A, G^{-1} A_x \rangle dx \tag{41}$$

where A is vector with components A_m that are vector fields defined for every smooth functional $R(u)$ via variational derivatives $A_m(R) = \delta R / \delta u_m$. Moreover this model is actually bi-Hamiltonian as there exist another invariant Poisson bivector

$$\hat{W} = \int_{-\infty}^{+\infty} \{ \langle C, A \rangle \langle U, A_x \rangle + \langle A_x, F A_x \rangle \} dx \tag{42}$$

that is compatible with W or in other words

$$[W, W] = [W, \hat{W}] = [\hat{W}, \hat{W}] = 0 \tag{43}$$

Corresponding Hamiltonians that produce Hamiltonian realization

$$\frac{d}{dt} U = \hat{W}(d\hat{H} \wedge dU) = W(dH \wedge dU) \tag{44}$$

of the evolution equations (37) are

$$\hat{H} = \frac{1}{2} \int_{-\infty}^{+\infty} \langle U, GU \rangle dx \tag{45}$$

and

$$H = \frac{1}{2} \int_{-\infty}^{+\infty} \{ \langle C, GU \rangle \langle U, GU \rangle + 2 \langle F G U_x, GU \rangle \} dx \tag{46}$$

The most remarkable property of system (37) is that it possesses set of conservation laws that belong to single orbit obtained from centre of Poisson

algebra via one-parameter group of transformations generated by the following vector field

$$
\begin{aligned}
&E=\int_{-\infty}^{+\infty}\{\langle C,GU\rangle\langle U,A_x\rangle+\langle U,GU\rangle\langle C,A_x\rangle\\
&+\langle U,GU_x\rangle\langle C,A\rangle+2\langle FGU,A_{xx}\rangle\}xdx\\
&=\int_{-\infty}^{+\infty}\{\langle C,GU\rangle\langle U,A\rangle+\langle U,GU\rangle\langle C,A\rangle+4\langle FGU_x,A\rangle\\
&+x(\langle C,GU_x\rangle\langle U,A\rangle+\langle C,GU\rangle\langle U_x,A\rangle\\
&+\langle U,GU_x\rangle\langle C,A\rangle+2\langle FGU_{xx},A\rangle)\}dx
\end{aligned}
\tag{47}
$$

Note that centre of Poisson algebra (with respect to bracket defined by W) is formed by functionals of the following type

$$
J=\int_{-\infty}^{+\infty}\langle K,U\rangle dx \tag{48}
$$

where K is arbitrary constant vector and applying group of transformations generated by E to this functional J yields the infinite sequence of functionals

$$
\begin{aligned}
&J^{(0)}=\int_{-\infty}^{+\infty}\langle K,U\rangle dx\\
&J^{(1)}=L_E J^{(0)}=\frac{1}{2}\langle C,K\rangle\int_{-\infty}^{+\infty}\langle U,GU\rangle dx\\
&J^{(2)}=(L_E)^2 J^{(0)}=\langle C,K\rangle\int_{-\infty}^{+\infty}\{\langle C,GU\rangle\langle U,GU\rangle+2\langle FGU_x,GU\rangle\}dx\\
&J^{(3)}=(L_E)^3 J^{(0)}=\frac{1}{4}\langle C,K\rangle\int_{-\infty}^{+\infty}\{3\langle C,GC\rangle\langle U,GU\rangle^2\\
&+12\langle C,GU\rangle^2\langle U,GU\rangle+32\langle C,GU\rangle\langle GU,FGU_x\rangle\\
&+24\langle U,GU\rangle\langle GC,FGU_x\rangle+48\langle FGU_x,GFGU_x\rangle\}dx\\
&J^{(m)}=(L_E)^m J^{(0)}=L_E J^{(m-1)}
\end{aligned}
\tag{49}
$$

One can check that the vector field E satisfies condition

$$
[E,[E,W]]=0 \tag{50}
$$

and according to Theorem 2 the sequence $J^{(m)}$ is involutive. So $J^{(m)}$ are conservation laws of bi-Hamiltonian dynamical system (37) and vector field E

is related to non-Noether symmetries of evolutionary equations (see Remark 1).
Note that in special case when C, F, G, K have the following form

$$D = 2, \qquad F_{12} = -F_{21} = \frac{1}{2}c, \qquad C = K = (0, 1), \qquad G = 1 \tag{51}$$

model (37) reduces to modified Boussinesq system discussed above. Another choice of constants C, F, G, K

$$\begin{aligned} D = 2, \qquad F_{12} = -F_{21} = \frac{1}{2}c, \qquad C = K = (0, 1) \\ G_{12} = G_{21} = 1, \qquad G_{11} = G_{22} = 0 \end{aligned} \tag{52}$$

gives rise to Broer-Kaup system described in previous sample.
Conclusions. Groups of transformations of Poisson manifold that possess involutive orbits play important role in some integrable models where conservation laws form orbit of non-Noether symmetry group. Therefore classification of vector fields that generate such a groups would create good background for description of remarkable class of integrable system that have interesting geometric origin. The present paper is an attempt to outline one particular class of vector fields that are related to non-Noether symmetries of Hamiltonian dynamical systems and produce involutive families of conservation laws.
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